Associated graded rings of one-dimensional analytically irreducible rings II

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Abstract

Lance Bryant noticed in his thesis [3], that there was a flaw in our paper [2]. It can be fixed by adding a condition, called the BF condition in [3]. We discuss some equivalent conditions, and show that they are fulfilled for some classes of rings, in particular for our motivating example of semigroup rings. Furthermore we discuss the connection to a similar result, stated in more generality, by Cortadella-Zarzuela in [4]. Finally we use our result to conclude when a semigroup ring in embedding dimension at most three has an associated graded which is a complete intersection.

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1 The BF condition

Let (R, m) be an equicharacteristic analytically irreducible and residually rational local 1-dimensional domain of embedding dimension ν, multiplicity e and residue field k. For the problems we study we may, and will, without loss of generality suppose that R is complete. So our hypotheses are equivalent to supposing R is a subring of $k[[t]]$ with $(R : k[[t]]) \neq 0$. Since $k[[t]]$, the integral closure of R, is a DVR, every nonzero element of R has a value, and we let $S = v(R) = \{v(r); r \in R, r \neq 0\}$. We denote by $w_0, \ldots, w_{e-1}$ the Apery set of $v(R)$ with respect to e, i.e., the set of smallest values in $v(R)$ in each congruence class $\pmod e$, and we assume $w_j \equiv j \pmod e$.

If $x \in R$ is an element of smallest positive value, i.e. $v(x) = e$, then xR is a minimal reduction of the maximal ideal, i.e. $m^{n+1} = xm^n$, for $n >> 0$. Conversely each minimal reduction of the maximal ideal is a principal ideal generated by an element x of value e. The smallest integer n such that $m^{n+1} = xm^n$ is called the reduction number and we denote it by r.

Observe that, if $v(x) = e$, then $\mathrm{Ap}_e(S) = S\setminus(e+S) = v(R)\setminus v(xR)$, therefore $w_j \notin v(xR)$, for $j = 0, \dots, e-1$.

Consider the m-adic filtration $m \supset m^2 \supset m^3 \supset \dots$. If $a \in R$, we set $\mathrm{ord}(a) := \max\{i \mid a \in m^i\}$. If $s \in S$, we consider the semigroup filtration $v(m) \supset v(m^2) \supset \dots$ and set $\mathrm{vord}(s) := \max\{i \mid s \in v(m^i)\}$. If $a \in m^i$, then $v(a) \in v(m^i)$ and so $\mathrm{ord}(a) \leq \mathrm{vord}(v(a))$.

According to [3], we say that the m-adic filtration is *essentially divisible with respect to the minimal reduction* xR if, whenever $u \in v(xR)$, then there is an $a \in xR$ with $v(a) = u$ and $\mathrm{ord}(a) = \mathrm{vord}(u)$. The m-adic filtration is *essentially divisible* if there exists a minimal reduction xR such that it is essentially divisible with respect to xR.

We fix for all the paper the following notation. Set, for $j = 0, \dots, e-1$, $b_j = \max\{i | w_j \in v(m^i)\}$, and let $c_j = \max\{i | w_j \in v(m^i + xR)\}$. Note that the numbers b_j's do not depend on the minimal reduction xR, on the contrary the c_j's depend on xR.

Lemma 1.1 *If I and J are ideals of R, then $v(I+J) = v(I) \cup v(J)$ is equivalent to $v(I \cap J) = v(I) \cap v(J)$.*

Proof. Let $V = v(I+J) \setminus v(I \cap J)$. Then

$$V = (v(I) \setminus v(I \cap J)) \cup (v(I+J) \setminus v(I)) = (v(J) \setminus v(I \cap J)) \cup (v(I+J) \setminus v(J)),$$

and both unions are disjoint. Since $(I+J)/J \simeq I/I \cap J$, we get that $|v(I+J) \setminus v(J)| = |v(I) \setminus v(I \cap J)|$. Thus

$$|v(I) \setminus v(I \cap J)| + |v(J) \setminus v(I \cap J)| = |(v(I) \cup v(J)) \setminus v(I \cap J)|$$

equals

$$|v(I+J) \setminus v(I)| + |v(I+J) \setminus v(J)| = |v(I+J) \setminus (v(I) \cap v(J))|.$$

Hence $|v(I) \cup v(J)| = |v(I+J)|$ if and only if $|v(I \cap J)| = |v(I) \cap v(J)|$. Since $v(I) \cup v(J) \subseteq v(I+J)$ and $v(I \cap J) \subseteq v(I) \cap v(J)$, we get the claim. $\square$

Proposition 1.2 *Let xR be a minimal reduction of m. Then the following conditions are equivalent:*

(1) *The m-adic filtration is essentially divisible with respect to xR.*
(2) *$v(m^i \cap xR) = v(m^i) \cap v(xR)$, for all $i \geq 0$.*
(3) *$v(m^i + xR) = v(m^i) \cup v(xR)$ for all $i \geq 0$.*
(4) *$b_j = c_j$ for $j = 0, \dots, e-1$.*

Proof. (1)$\Rightarrow$(2): Let $i \geq 0$ and $u \in v(m^i) \cap v(xR)$. Then $u \in v(xR)$ and $\mathrm{vord}(u) \geq i$. By (1) there exists $a \in xR$ with $v(a) = u$ and $\mathrm{ord}(a) = \mathrm{vord}(u)$. Thus $a \in m^i \cap xR$ and so $v(m^i \cap xR) \supseteq v(m^i) \cap v(xR)$. Since the other inclusion is trivial, we get an equality.

(2)$\Rightarrow$(1): If $u \in v(xR)$ and $\mathrm{vord}(u) = i$, then $u \in v(m^i) \cap v(xR)$, and by (2), $u \in v(m^i \cap xR)$. So there is $a \in m^i \cap xR$ with $v(a) = u$. For such a, $i \leq \mathrm{ord}(a) \leq \mathrm{vord}(u) = i$, and so $\mathrm{ord}(a) = i$.

That (2) and (3) are equivalent follows from Lemma 1.1 with $I = m^i$ and $J = xR$.

(3)$\Rightarrow$(4): Since $m^i \subseteq m^i + xR$, we have $v(m^i) \subseteq v(m^i + xR)$, so $b_j \leq c_j$. Suppose that $b_j < c_j$ for some j. Then $w_j \in v(m^{c_j} + xR) \setminus v(m^{c_j})$. Since $w_j \notin v(xR)$, we get that $v(m^{c_j}) \cup v(xR)$ is strictly included in $v(m^{c_j} + xR)$.
(4)$\Rightarrow$(3): If $u \in v(m^i + xR) \setminus v(xR)$, then $u \in v(R) \setminus v(xR) = \mathrm{Ap}_e v(R)$, so $u = w_j$ for some j. Then $w_j \in v(m^i + xR) \setminus v(m^i)$, so $b_j < c_j$. $\square$

Observe that if $R = k[[t^{n_1}, \ldots, t^{n_\nu}]]$ is a semigroup k-algebra and I, J are ideals generated by monomials, then $v(I \cap J) = v(I) \cap v(J)$ (and $v(I + J) = v(I) \cup v(J)$). This follows from the fact that if $I = (t^{i_1}, \ldots, t^{i_k})$ is generated by monomials, then $v(I) = \langle i_1, \ldots, i_k \rangle$. So, if we choose for the maximal ideal of R a monomial minimal reduction, by Proposition 1.2 we have that the m-adic filtration is essentially divisible with respect to such a reduction. If we choose a different minimal reduction this is not always the case, as the following example shows.

Example Let $R = k[[t^6, t^7, t^{15}]]$. By what we observed above, the m-adic filtration is essentially divisible with respect to the minimal reduction $t^6 R$. On the contrary, it is not essentially divisible with respect to the minimal reduction $(t^6 + t^7)R$, because $v(m^3 + (t^6 + t^7)R) \nsubseteq v(m^3) \cup v((t^6 + t^7)R)$ and we can apply Proposition 1.2 (3). As a matter of fact, $t^{21} - (t^6 + t^7)t^{15} \in m^3 + (t^6 + t^7)R$, thus $22 \in v(m^3 + (t^6 + t^7)R)$, but $22 \notin v(m^3) \cup v((t^6 + t^7)R)$.

This example shows also that the numbers c_j's depend on the minimal reduction. Considering $w_4 = 22$, with respect to the minimal reduction $t^6 R$, we get $b_4 = c_4 = 2$, but with respect to $(t^6 + t^7)R$, we get $2 = b_4 < c_4 = 3$.

In [2], we called a set $f_0, \ldots, f_{e-1}$ of elements of R an *Apery basis* if $v(f_j) \equiv j \pmod e$ and $\mathrm{ord}(f_j) = b_j$, for all j, $j = 0, \ldots, e-1$ and claimed that for all $i \geq 0$, m^i is a free W-module generated by elements of the form $x^{h_j} f_j$, where xR is a minimal reduction of m and $W = k[[x]]$. In [3] Lance Bryant showed that this is not always true, considering the example $R = k[[t^6, t^8 + t^9, t^{19}]]$ with $\mathrm{char}(k) = 0$. Here $e = 6$ and $v(R)$ has Apery set $0, 8, 16, 19, 27, 29$. Setting: $x = t^6, W = k[[t^6]]$ and $f_0 = 1, f_1 = t^8 + t^9, f_2 = t^{16} + 2t^{17} + t^{18}, f_3 = t^{19}, f_4 = t^{27} + t^{28}, f_5 = t^{29}$ he gets $m^3 = x^3 f_0 W + x^2 f_1 W + x f_2 W + gW + x f_4 W + x f_5 W$ where $g = (t^8 + t^9)^3 - (t^6)^4 = 3t^{25} + 3t^{26} + t^{27} \in m^3$. On the other hand $x^h f_3 = t^6 t^{19} = t^{25} \in m^2 \setminus m^3$.

According to [3], we say that the m-adic filtration satisfies the BF *condition* if there exists a minimal reduction xR of m and a set of elements $\{f_0, \ldots, f_{e-1}\}$

of R with $v(f_j) = w_j$ such that each power of m is a free $k[[x]]$-module generated by elements of the form $x^{h_j} f_j$.

The BF condition depends on the choice of the elements $\{f_0, \dots, f_{e-1}\}$ and on the reduction. In [2] we noted that, if $R = k[[t^4, t^6+t^7, t^{13}]]$, with $\mathrm{char}(k) \neq 2$, then $\mathrm{Ap}_4(v(R)) = \{0, 6, 13, 15\}$ and setting $f_0 = 1,\ f_1 = t^6 + t^7,\ f_2 = 2t^{13} + t^{14},\ f_3 = t^{15}$, $x = t^4$, $W = k[[t^4]]$, we get that each power of the maximal ideal is a free W-module generated by elements of the form $x^{h_j} f_j$. For example:

$$m = xf_0W + f_1W + f_2W + f_3W$$

$$m^2 = x^2 f_0W + xf_1W + f_2W + xf_3W$$

$$m^3 = xm^2 = x^3 f_0W + x^2 f_1W + xf_2W + xf_3W$$

If we replace f_2 with t^{13}, since $t^{13} \in m \setminus m^2$, we don't have the free basis of the requested form for m^2. Thus this example shows that the BF condition depends on the choice of the elements $\{f_0, \dots, f_{e-1}\}$. To show that the BF condition depends on the reduction, we can consider the example above, $R = k[[t^6, t^7, t^{15}]]$. We get that $f_0 = 0, f_1 = t^7, f_2 = t^{14}, f_3 = t^{15}, f_4 = t^{22}, f_5 = t^{29}$ is an Apery basis but, choosing the minimal reduction $xR = (t^6 + t^7)R$, m^4 is not a free $k[[x]]$-module generated by elements of the form $x^{h_j} f_j$, because $\mathrm{Ap}_6(v(m^4)) = \{24, 25, 26, 27, 28, 35\}$ and an element of the form $x^{h_j} f_j$ of value 28 is $(t^6 + t^7)t^{22}$, which is not in m^4.

Proposition 1.3 *Let $W = k[[x]]$, where xR is a minimal reduction of m and let $f_0, \dots, f_{e-1}$ be elements of R with $v(f_j) \equiv j \pmod e$. Then the following conditions are equivalent:*
(1) *For all $i \geq 0$, m^i is a free W-module generated by elements of the form $x^{h_j} f_j$.*
(2) *For all $i \geq 0$, $\mathrm{Ap}_e(v(m^i)) = \{v(x^{h_j} f_j)\}$ for some $x^{h_j} f_j \in m^i$, $j = 0, \dots, e-1$.*
(3) *If $\sum_{j=0}^{e-1} d_j(x) f_j \in m^i$ with $d_j(x) \in W$ for all j, then $d_j(x) f_j \in m^i$ for each j.*

Proof. (1)⇒(3): Let $a = \sum_{j=0}^{e-1} d_j(x) f_j \in m^i$. Since $\{x^{h_j} f_j\}$ is a free basis for m^i, we also have $a = \sum_{j=0}^{e-1} d'_j(x) x^{h_j} f_j$ for some $d'_j(x)$, and $d_j(x) = d'_j(x) x^{h_j}$. Now $x^{h_j} f_j \in m^i$, so $d_j(x) f_j \in m^i$.
(3)⇒(2): Let $u \in \mathrm{Ap}_e(v(m^i))$, so $u = v(a)$ for some $a \in m^i$. We have $a = \sum_{j=0}^{e-1} d_j(x) f_j$, with $d_j(x) f_j \in m^i$ for all j. Let $v(a) \equiv v(f_j) \pmod e$. Then $v(a) = v(d_j(x) f_j)$. Let $d_j(x) = \sum_{i \geq l} k_i x^i$, with $k_i \in k, k_l \neq 0$. Then we claim that $\mathrm{ord}(d_j(x) f_j) = \mathrm{ord}(x^l f_j)$. Suppose that $x^l f_j \in m^h \setminus m^{h+1}$. Then $d_j(x) f_j \in m^h$ since all summands do. If $d_j(x) f_j \in m^{h+1}$, then $k_l x^l f_j = d_j(x) f_j - \sum_{i \geq l+1} k_i x^i f_j \in m^{h+1}$, a contradiction. Thus $v(a) = v(x^l f_j)$, $x^l f_j \in m^i$.
(2)⇒(1): By Lemma 2.1 (1) of [2]. □

Proposition 1.4 *If the m-adic filtration satisfies the BF condition, it is essentially divisible.*

Proof. Let xR be a minimal reduction of m and let $f_0, \ldots, f_{e-1}$ be elements in R satisfying the BF condition, i.e. condition (2) in Proposition 1.3. We claim that condition (2) in Proposition 1.2 is satisfied. Let $v \in v(m^i) \cap v(xR)$, $v = v_j + le$, with $v_j \in \mathrm{Ap}_e(v(m^i))$, for some $l \geq 0$. We have $v_j = v(x^{h_j} f_j)$, for some j. Thus $x^{h_j+l} f_j \in m^i \cap xR$ and $v(x^{h_j+l} f_j) = v$. Note that $h_j + l > 0$. $\square$

There are several cases in which the BF condition holds.

Proposition 1.5 *The BF-condition holds for the m-adic filtration in each of the following cases:*
(1) *R is a semigroup k-algebra.*
(2) *The reduction number r is at most 2.*
(3) *The embedding dimension ν is at most 2.*

Proof. (1): Let $R = k[[t^{n_1}, \ldots, t^{n_\nu}]]$ and $\mathrm{Ap}(v(R)) = \{w_0, \ldots, w_{e-1}\}$. Choosing the monomial Apery basis $f_j = t^{w_j}$, for $j = 0, \ldots, e-1$ and the monomial minimal reduction $xR = t^{n_1}R = t^e R$, if $\mathrm{Ap}(v(m^i)) = \{w_0 + h_0 e, \ldots, w_{e-1} + h_{e-1}e\}$, then m^i is a free $k[[t^e]]$-module generated by $t^{eh_j} f_j = t^{h_j e + w_j}$.
(2): Let xR is a minimal reduction of m and let $f_0, \ldots, f_{e-1}$ be an Apery basis of R. Then the Apery sets of $v(m^i))$, with $i \leq 2$ can always be realized as in Proposition 1.3 (2). In fact, for $v(m^2)$, note that $v(x^2 f_0) = 2e \in \mathrm{Ap}(v(m^2))$. Moreover, if $f_j \in m \setminus m^2$, then $v(xf_j) \in \mathrm{Ap}(v(m^2))$ and if $f_j \in m^2$, then $v(f_j) \in \mathrm{Ap}(v(m^2))$. If $i \geq 2$, then $m^{i+1} = xm^i$, which gives the claim.
(3) In the plane case, setting $m = \langle x, y \rangle$, using the Weierstrass Preparation Theorem, we noted in [1, Section 2] that R is a W-module generated by $1, y, y^2, ...,$ y^{e-1} and replacing each y^j with a suitable $y_j = y^j + \phi(x,y)$ ($\phi(x,y) \in m^j$), we get an Apery basis for R. Consider a power m^i of the maximal ideal. Using the above observation, m^i is generated as W-module by $x^i, x^{i-1}y, x^{i-2}y^2, \ldots, y^i, y^{i+1},$ $\ldots, y^{i(e-1)}$. Now working on the powers y^j as we do in [1], we can modify the generators, getting the e elements $x^i, x^{i-1}y, x^{i-2}y_2, \ldots, y_{e-1}$, which are still in m^i, are of the requested form and such that their values form an Apery set for $v(m^i)$. $\square$

Example Consider $R = \mathbb{C}[[t^6, t^8 + t^9]]$. Setting $x = t^6$, $y = t^8 + t^9$, as in [1], we can see that an Apery basis for R is $1, y, y_2 = y^2, y_3 = y^3 - x^4 = 3t^{25} + ..., y_4 = y^4 - x^4 y = 5t^{33} + ..., y_5 = y^5 - x^4 y^2 = 5t^{41} + ...$ Considering for example m^3, we see it is a free W-module generated by $x^3, x^2y, xy_2, y_3, y_4, y_5$.

2 The associated graded ring

Let $\mathrm{gr}(R)$ be the associated graded ring with respect to the m-adic filtration, $\mathrm{gr}(R) = \bigoplus_{i \geq 0} m^i/m^{i+1}$. The CM-ness of $\mathrm{gr}(R)$ is equivalent to the existence of a nonzerodivisor in the homogeneous maximal ideal. If such a nonzerodivisor exists, then x^*, the image of x in $\mathrm{gr}(R)$ (where x is any element of value e) is a nonzerodivisor. We fix this notation and denote by

$\mathrm{Hilb}_R(z) = \sum_{i\geq 0} l_R(m^i/m^{i+1})z^i$ the Hilbert series of R and by $\mathrm{Hilb}_{R/xR}(z) = \sum_{i\geq 0} l_R(m^i + xR/m^{i+1} + xR)z^i$ the Hilbert series of R/xR. Recall that

$$(1-z)\mathrm{Hilb}_R(z) \leq \mathrm{Hilb}_{R/xR}(z)$$

and the equality holds if and only if $\mathrm{gr}(R)$ is CM (cf. e.g. [3] or [4]).

We start noting that, if $\mathrm{gr}(R)$ is CM, then the conditions analyzed in the previous section are equivalent.

Proposition 2.1 *If* $\mathrm{gr}(R)$ *is CM, then the* m*-adic filtration is essentially divisible if and only if it satisfies the BF condition.*

Proof. Suppose that the m-adic filtraion is essentially divisible with respect to xR. We claim that there exist $f_0, \ldots, f_{e-1}$ in R satifying condition (2) of Proposition 1.3. If $n \geq r$, where r is the reduction number, then $m^n \subseteq xR$. Thus, if $u \in \mathrm{Ap}_e(v(m^n))$, $u \equiv j \pmod e$, then there exist $a \in R$, $a = xa'$, with $v(a) = u$ and $\mathrm{ord}(a) = n$. We have $v(a') = u - e$ and $\mathrm{ord}(a') = \mathrm{ord}(a) - 1$, because $\mathrm{gr}(R)$ is CM. Now there are two possibilities. If $v(a') \notin v(xR)$, i.e. $v(a') = w_j$, we choose $f_j = a'$. If $v(a') \in v(xR)$, then, since R is essentially divisible, there exist $b \in xR$, $b = xb'$, with $v(b) = v(a')$ and $\mathrm{ord}(b) = \mathrm{ord}(a')$. Moreover $b \in \mathrm{Ap}(v(m^{n-1}))$, because otherwise $u - 2e \in v(m^{n-1})$ and $u - e \in v(m^n)$, a contradiction. Continuing in this way we arrive to get the element f_j requested.

We denote by R' the first neighborhood ring or the blowup of R, i.e. the overring $\bigcup_{n\geq 0}(m^n : m^n)$. It is well known that, if $v(x) = e$, $R' = R[x^{-1}m] = \bigcup_{i\geq 0}\{yx^{-i}; y \in m^i\}$, cf. [8]. Let $w'_0, \ldots, w'_{e-1}$ be the Apery set of $v(R')$ with respect to e, with $w'_j \equiv j \pmod e$. For each j, $j = 0, \ldots, e-1$, define as in [2] a_j by $w'_j = w_j - a_j e$.

If $f_j \in m^i$, then $f_j x^{-i} \in R'$, so $v(f_j x^{-i}) = w_j - ie \in v(R')$. It follows that $w_j - b_j e \in v(R')$. Since $w'_j = w_j - a_j e$ is the smallest in $v(R')$, in its congruence class $\pmod e$, we have that $a_j \geq b_j$, for $j = 0, \ldots, e-1$.

In [2, Theorem 2.6] we stated the following: The ring $\mathrm{gr}(R)$ is CM if and only if $a_j = b_j$, for $j = 0, \ldots, e-1$.

As Lance Bryant pointed out, the proof of that theorem given in [2] works under the assumption that the m-adic filtration satisfies the BF condition.

Theorem 2.2 *If* R *satifies the BF condition then* $\mathrm{gr}(R)$ *is CM if and only if* $a_j = b_j$*, for* $j = 0, \ldots, e-1$*.*

Proof. If the BF condition is satisfied, the proof given in [2] holds.

In [4] T. Cortadellas and S. Zarzuela proved, in more general hypotheses for R, a criterion for the CM-ness of $\mathrm{gr}(R)$. They consider the microinvariants of J. Elias, i.e. the numbers ϵ_j which appear in the decomposition of the torsion module

$$R'/R = \bigoplus_{j=0}^{e-1} W/x^{\epsilon_j}W$$

where R' is the blowup, xR a minimal reduction of m and $W = k[[x]]$. With our hypotheses and notation, they show in particular that $\mathrm{gr}(R)$ is CM if and only if $c_j = \epsilon_j$, for $j = 0, \ldots, e-1$, [4, Theorem 4.2]. Comparing their result with ours, we see that they are coherent but different. In fact, if the m-adic filtration satisfies the BF condition, then, for $j = 0, \ldots, e-1$, $\epsilon_j = a_j$ by [2, Proposition 2.5] and $b_j = c_j$ by Propositions 1.2 and 1.4, so their result coincide with ours. The hypotheses on the ring in their result are more general, but the numbers c_j's depend on the minimal reduction. On the other hand, the numbers a_j's and b_j's which we consider do not depend on the minimal reduction and in our criterion the CM-ness of $\mathrm{gr}(R)$ can be read off just looking at the semigroup filtration $v(m^0) \supset v(m) \supset v(m^2) \supset \ldots$. As a matter of fact, since $R' = x^{-n}m^n$, for $n >> 0$, $v(R') = v(m^n) - ne$, for $n >> 0$, so the a_j's which relate the Apery sets of $v(R)$ and $v(R')$, can be read in the semigroup filtration $\{v(m^i)\}_{i \geq 0}$.

We give now some applications. Given an analytically irreducible ring satisfying our hypotheses, we denote by $a_j(R)$ and $b_j(R)$ the numbers defined above.

Proposition 2.3 *Let R and T be rings satifying the BF condition, with the same multiplicity e and with $a_j(R) = a_j(T)$, $b_j(R) = b_j(T)$, for $j = 0, \ldots, e-1$. If $\mathrm{gr}(R)$ is CM, then also $gr(T)$ is CM and R and T have the same Hilbert series.*

Proof. Since $\mathrm{gr}(R)$ is CM, by Theorem 2.2, $a_j(R) = b_j(R)$, for $j = 0, \ldots, e-1$. So also $a_j(T) = b_j(T)$, for $j = 0, \ldots, e-1$ and $\mathrm{gr}(T)$ is CM. If xR (respectively yT) is a minimal reduction of the maximal ideal of R (respectively of T), then, since $b_j(R) = c_j(R)$ and $b_j(T) = c_j(T)$ (cf. Proposition 1.2), the Hilbert series of R/xR and T/yT are the same. Since $\mathrm{Hilb}_{R/xR}(z) = (1-z)\mathrm{Hilb}_R(z)$ and $\mathrm{Hilb}_{T/yT}(z) = (1-z)\mathrm{Hilb}_R(z)$, also the Hilbert series of R and T are the same. $\square$

Sometimes we can use the BF condition to draw conclusions about when $\mathrm{gr}(R)$ is a complete intersection (CI). We will use that if $x \in R$ is a nonzerodivisor in R such that x^* is a nonzerodivisor in $\mathrm{gr}(R)$, then $\mathrm{gr}(R/xR) = \mathrm{gr}(R)/(x^*)$, [7, Lemma(b)].

Example If $R = k[[X,Y]]/(f)$ is a plane branch, then $\mathrm{gr}(R) = k[X,Y]/(f^*)$, where f^* is the image of f in $\mathrm{gr}(R)$, so $\mathrm{gr}(R)$ is a complete intersection. The semigroups S for which $k[[S]]$ is a CI were determined in [5]. If $\mathrm{gr}(k[[S]])$ is a CI, then necessarily $k[[S]]$ is a CI [9, Corollary 2.4]. If S is generated by three elements and is a CI, the generators are of the form $na, nb, n_1a + n_2b$, $a < b$, [6] or (with an easier proof) [10, Lemma 1]. Then

$$k[[S]] = k[[X,Y,Z]]/(X^b - Y^a, Z^n - X^{n_1}Y^{n_2})$$

It is determined in [7] when $\mathrm{gr}_m(k[[S]])$ is a CI when S is 3-generated. The result is

a) $S = \langle na, nb, n_1a \rangle$.

b) $S=\langle na,nb,n_1a+n_2b\rangle$, $na<n_1a+n_2b<nb$, $n\le n_1+n_2$.
c) $S=\langle na,nb,n_1a+n_2b\rangle$, $na<nb<n_1a+n_2b$, $n\le n_1+n_2$.
Let $x=t^{na},y=t^{nb},z=t^{n_1a+n_2b}$.

In case a), if $n<n_1$, $\mathrm{gr}(k[[S]]/(x))\cong k[Y,Z]/(Y^a,Z^n)$. An Apery basis for $k[[S]]$ is $\{y^iz^j;0\le i<a,0\le j<n\}$. Suppose $R=k[[t^{na},g_2,g_3]]$ with $v(g_2)=nb,v(g_3)=n_1a$, and that $\{g_2^ig_3^j;0\le i<a,0\le j<n\}$ is an Apery basis for R, and that R satisfies the BF condition. Then $x=t^{na}$ is a minimal reduction also of the maximal ideal of R, and the a_j's and b_j's are the same for $k[[S]]$ and R, so $\mathrm{gr}(R)$ is CM, and in particular x^* is a nonzerodivisor in $\mathrm{gr}(R)$. We have that $\mathrm{gr}(R)$ is a CI if and only if $\mathrm{gr}(R/xR)=\mathrm{gr}(R)/(x^*)$ is a CI. Since $v(g_2^ig_3^j)\notin v(xR)$ if $0\le i<a,0\le j<n$, and they all have values in different congruence classes (mod $v(x)$), we get that $\mathrm{gr}(R)/(x^*)\cong \mathrm{gr}(k[[S]])/(x^*)\cong k[Y,Z]/(Y^a,Z^n)$. Thus $\mathrm{gr}(R)$ is a CI. A concrete example is $R=k[[t^6,t^8+ct^{13}+dt^{19},t^9]]$, $c,d\in k$.

If $n_1<n$, then $\mathrm{gr}(k[[S]]/(z))=k[X,Y]/(Y^a,X^{n_1})$, and $\{y^ix^j;0\le i<a,0\le j<n_1\}$ is an Apery basis for $k[[S]]$. Suppose $R=k[[t^{n_1a},g_2,g_3]]$ with $v(g_2)=na,v(g_3)=nb$, and that $\{g_3^ig_2^j;0\le i<a,0\le j<n_1\}$ is an Apery basis for R, and that R satisfies the BF condition. As above we get that $\mathrm{gr}(R)$ is a CI. A concrete example is $k[[t^6,t^9+ct^{11},t^4]]$, $c\in k$.

In case b) an Apery set is $\{y^iz^j;0\le i<a,0\le j<n\}$. Suppose $R=k[[t^{na},g_2,g_3]]$, $v(g_2)=n_1a+n_2b,v(g_3)=nb$, and that $\{g_3^ig_2^j;0\le i<a,0\le j<n\}$ is an Apery set for R, and that R satifies the BF condition. Reasoning as above, we get that $\mathrm{gr}(R)$ is a CI. A concrete example is $k[[t^6,t^7+ct^{11},t^9]]$, $c\in k$.

In case c) an Apery set is $\{y^iz^j;0\le i<a,0\le j<n\}$. Suppose $R=k[[t^{na},g_2,g_3]]$, $v(g_2)=nb,v(g_3)=n_1a+n_2b$, and that $\{g_2^ig_3^j;0\le i<a,0\le j<n\}$ is an Apery set for R, and that R satifies the BF condition. Reasoning as above, we get that $\mathrm{gr}(R)$ is a CI. A concrete example is $k[[t^4,t^6,t^7+ct^9]]$, $c\in k$.

We end with some questions:

1. Does the converse of Proposition 1.4 hold?

2. Is Theorem 2.2 true, without assuming the BF-condition?

3. Is always $\epsilon_j=a_j$, for $j=0,\dots,e-1$ without assuming the BF-condition?

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Associated graded rings of one-dimensional analytically irreducible rings II

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Abstract

Lance Bryant noticed in his thesis [3], that there was a flaw in our paper [2]. It can be fixed by adding a condition, called the BF condition in [3]. We discuss some equivalent conditions, and show that they are fulfilled for some classes of rings, in particular for our motivating example of semigroup rings. Furthermore we discuss the connection to a similar result, stated in more generality, by Cortadella-Zarzuela in [4]. Finally we use our result to conclude when a semigroup ring in embedding dimension at most three has an associated graded which is a complete intersection.

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1 The BF condition

Let (R, m) be an equicharacteristic analytically irreducible and residually rational local 1-dimensional domain of embedding dimension ν, multiplicity e and residue field k. For the problems we study we may, and will, without loss of generality suppose that R is complete. So our hypotheses are equivalent to supposing R is a subring of $k[[t]]$ with $(R : k[[t]]) \neq 0$. Since $k[[t]]$, the integral closure of R, is a DVR, every nonzero element of R has a value, and we let $S = v(R) = \{v(r); r \in R, r \neq 0\}$. We denote by $w_0, \ldots, w_{e-1}$ the Apery set of $v(R)$ with respect to e, i.e., the set of smallest values in $v(R)$ in each congruence class $\pmod e$, and we assume $w_j \equiv j \pmod e$.

If $x \in R$ is an element of smallest positive value, i.e. $v(x) = e$, then xR is a minimal reduction of the maximal ideal, i.e. $m^{n+1} = xm^n$, for $n >> 0$. Conversely each minimal reduction of the maximal ideal is a principal ideal generated by an element x of value e. The smallest integer n such that $m^{n+1} = xm^n$ is called the reduction number and we denote it by r.

Observe that, if $v(x) = e$, then $\mathrm{Ap}_e(S) = S \setminus (e+S) = v(R) \setminus v(xR)$, therefore $w_j \notin v(xR)$, for $j = 0, \ldots, e-1$.

Consider the m-adic filtration $m \supset m^2 \supset m^3 \supset \ldots$. If $a \in R$, we set $\mathrm{ord}(a) := \max\{i \mid a \in m^i\}$. If $s \in S$, we consider the semigroup filtration $v(m) \supset v(m^2) \supset \ldots$ and set $\mathrm{vord}(s) := \max\{i \mid s \in v(m^i)\}$. If $a \in m^i$, then $v(a) \in v(m^i)$ and so $\mathrm{ord}(a) \leq \mathrm{vord}(v(a))$.

According to [3], we say that the m-adic filtration is *essentially divisible with respect to the minimal reduction* xR if, whenever $u \in v(xR)$, then there is an $a \in xR$ with $v(a) = u$ and $\mathrm{ord}(a) = \mathrm{vord}(u)$. The m-adic filtration is *essentially divisible* if there exists a minimal reduction xR such that it is essentially divisible with respect to xR.

We fix for all the paper the following notation. Set, for $j = 0, \ldots, e-1$, $b_j = \max\{i | w_j \in v(m^i)\}$, and let $c_j = \max\{i | w_j \in v(m^i + xR)\}$. Note that the numbers b_j's do not depend on the minimal reduction xR, on the contrary the c_j's depend on xR.

Lemma 1.1 *If I and J are ideals of R, then $v(I+J) = v(I) \cup v(J)$ is equivalent to $v(I \cap J) = v(I) \cap v(J)$.*

Proof. Let $V = v(I+J) \setminus v(I \cap J)$. Then

$$V = (v(I) \setminus v(I \cap J)) \cup (v(I+J) \setminus v(I)) = (v(J) \setminus v(I \cap J)) \cup (v(I+J) \setminus v(J))$$

and both unions are disjoint. Since $(I+J)/J \simeq I/I \cap J$, we get that $|v(I+J) \setminus v(J)| = |v(I) \setminus v(I \cap J)|$ and similarly that $|v(I+J) \setminus v(I)| = |v(J) \setminus v(I \cap J)|$. Suppose that $v(I \cap J) \subsetneq v(I) \cap v(J)$, i.e. that there is a value $v_0 \in (v(I) \setminus v(I \cap J)) \cap (v(J) \setminus v(I \cap J))$. Thus $v_0 \notin (v(I+J) \setminus v(J))$ and by cardinality reasons also $(v(I+J) \setminus v(I)) \cap (v(I+J) \setminus v(J)) \neq \emptyset$, i.e. $v(I+J) \supsetneq v(I) \cup v(J)$. The other implication is symmetric and we get the claim. $\square$

Proposition 1.2 *Let xR be a minimal reduction of m. Then the following conditions are equivalent:*

(1) *The m-adic filtration is essentially divisible with respect to xR.*
(2) *$v(m^i \cap xR) = v(m^i) \cap v(xR)$, for all $i \geq 0$.*
(3) *$v(m^i + xR) = v(m^i) \cup v(xR)$ for all $i \geq 0$.*
(4) *$b_j = c_j$ for $j = 0, \ldots, e-1$.*

Proof. (1)$\Rightarrow$(2): Let $i \geq 0$ and $u \in v(m^i) \cap v(xR)$. Then $u \in v(xR)$ and $\mathrm{vord}(u) \geq i$. By (1) there exists $a \in xR$ with $v(a) = u$ and $\mathrm{ord}(a) = \mathrm{vord}(u)$. Thus $a \in m^i \cap xR$ and so $v(m^i \cap xR) \supseteq v(m^i) \cap v(xR)$. Since the other inclusion is trivial, we get an equality.

(2)⇒(1): If $u \in v(xR)$ and $\mathrm{vord}(u) = i$, then $u \in v(m^i) \cap v(xR)$, and by (2), $u \in v(m^i \cap xR)$. So there is $a \in m^i \cap xR$ with $v(a) = u$. For such a, $i \leq \mathrm{ord}(a) \leq \mathrm{vord}(u) = i$, and so $\mathrm{ord}(a) = i$.

That (2) and (3) are equivalent follows from Lemma 1.1 with $I = m^i$ and $J = xR$.

(3)⇒(4): Since $m^i \subseteq m^i + xR$, we have $v(m^i) \subseteq v(m^i + xR)$, so $b_j \leq c_j$. Suppose that $b_j < c_j$ for some j. Then $w_j \in v(m^{c_j} + xR) \setminus v(m^{c_j})$. Since $w_j \notin v(xR)$, we get that $v(m^{c_j}) \cup v(xR)$ is strictly included in $v(m^{c_j} + xR)$.
(4)⇒(3): If $u \in v(m^i + xR) \setminus v(xR)$, then $u \in v(R) \setminus v(xR) = \mathrm{Ap}_e v(R)$, so $u = w_j$ for some j. Then $w_j \in v(m^i + xR) \setminus v(m^i)$, so $b_j < c_j$. □

Observe that if $R = k[[t^{n_1}, \ldots, t^{n_\nu}]]$ is a semigroup k-algebra and I, J are ideals generated by monomials, then $v(I \cap J) = v(I) \cap v(J)$ (and $v(I + J) = v(I) \cup v(J)$). This follows from the fact that if $I = (t^{i_1}, \ldots, t^{i_k})$ is generated by monomials, then $v(I) = \langle i_1, \ldots, i_k \rangle$. So, if we choose for the maximal ideal of R a monomial minimal reduction, by Proposition 1.2 we have that the m-adic filtration is essentially divisible with respect to such a reduction. If we choose a different minimal reduction this is not always the case, as the following example shows.

Example Let $R = k[[t^6, t^7, t^{15}]]$. By what we observed above, the m-adic filtration is essentially divisible with respect to the minimal reduction t^6R. On the contrary, it is not essentially divisible with respect to the minimal reduction $(t^6 + t^7)R$, because $v(m^3 + (t^6 + t^7)R) \nsubseteq v(m^3) \cup v((t^6 + t^7)R)$ and we can apply Proposition 1.2 (3). As a matter of fact, $t^{21} - (t^6 + t^7)t^{15} \in m^3 + (t^6 + t^7)R$, thus $22 \in v(m^3 + (t^6 + t^7)R)$, but $22 \notin v(m^3) \cup v((t^6 + t^7)R)$.

This example shows also that the numbers c_j's depend on the minimal reduction. Considering $w_4 = 22$, with respect to the minimal reduction t^6R, we get $b_4 = c_4 = 2$, but with respect to $(t^6 + t^7)R$, we get $2 = b_4 < c_4 = 3$.

In [2], we called a set $f_0, \ldots, f_{e-1}$ of elements of R an *Apery basis* if $v(f_j) \equiv j \pmod{e}$ and $\mathrm{ord}(f_j) = b_j$, for all j, $j = 0, \ldots, e-1$ and claimed that for all $i \geq 0$, m^i is a free W-module generated by elements of the form $x^{h_j} f_j$, where xR is a minimal reduction of m and $W = k[[x]]$. In [3] Lance Bryant showed that this is not always true, considering the example $R = k[[t^6, t^8 + t^9, t^{19}]]$ with $\mathrm{char}(k) = 0$. Here $e = 6$ and $v(R)$ has Apery set $0, 8, 16, 19, 27, 29$. Setting: $x = t^6, W = k[[t^6]]$ and $f_0 = 1, f_1 = t^8 + t^9, f_2 = t^{16} + 2t^{17} + t^{18}, f_3 = t^{19}, f_4 = t^{27} + t^{28}, f_5 = t^{29}$ he gets $m^3 = x^3 f_0 W + x^2 f_1 W + x f_2 W + gW + x f_4 W + x f_5 W$ where $g = (t^8 + t^9)^3 - (t^6)^4 = 3t^{25} + 3t^{26} + t^{27} \in m^3$. On the other hand $x^h f_3 = t^6 t^{19} = t^{25} \in m^2 \setminus m^3$.

According to [3], we say that the m-adic filtration satisfies the BF *condition* if there exists a minimal reduction xR of m and a set of elements $\{f_0, \ldots, f_{e-1}\}$ of R with $v(f_j) = w_j$ such that each power of m is a free $k[[x]]$-module generated by elements of the form $x^{h_j} f_j$.

The BF condition depends on the choice of the elements $\{f_0, \ldots, f_{e-1}\}$ and on the reduction. In [2] we noted that, if $R = k[[t^4, t^6 + t^7, t^{13}]]$, with $\mathrm{char}(k) \neq 2$,

then $\mathrm{Ap}_4(v(R)) = \{0, 6, 13, 15\}$ and setting $f_0 = 1,\ f_1 = t^6 + t^7,\ f_2 = 2t^{13} + t^{14},\ f_3 = t^{15}$, $x = t^4$, $W = k[[t^4]]$, we get that each power of the maximal ideal is a free W-module generated by elements of the form $x^{h_j} f_j$. For example:

$$m = xf_0W + f_1W + f_2W + f_3W$$

$$m^2 = x^2 f_0W + xf_1W + f_2W + xf_3W$$

$$m^3 = xm^2 = x^3 f_0W + x^2 f_1W + xf_2W + xf_3W$$

If we replace f_2 with t^{13}, since $t^{13} \in m \setminus m^2$, we don't have the free basis of the requested form for m^2. Thus this example shows that the BF condition depends on the choice of the elements $\{f_0, \dots, f_{e-1}\}$. To show that the BF condition depends on the reduction, we can consider the example above, $R = k[[t^6, t^7, t^{15}]]$. We get that $f_0 = 0, f_1 = t^7, f_2 = t^{14}, f_3 = t^{15}, f_4 = t^{22}, f_5 = t^{29}$ is an Apery basis but, choosing the minimal reduction $xR = (t^6 + t^7)R$, m^4 is not a free $k[[x]]$-module generated by elements of the form $x^{h_j} f_j$, because $\mathrm{Ap}_6(v(m^4)) = \{24, 25, 26, 27, 28, 35\}$ and an element of the form $x^{h_j} f_j$ of value 28 is $(t^6 + t^7)t^{22}$, which is not in m^4.

Proposition 1.3 *Let $W = k[[x]]$, where xR is a minimal reduction of m and let $f_0, \dots, f_{e-1}$ be elements of R with $v(f_j) \equiv j \pmod{e}$. Then the following conditions are equivalent:*
(1) For all $i \geq 0$, m^i is a free W-module generated by elements of the form $x^{h_j} f_j$.
(2) For all $i \geq 0$, $\mathrm{Ap}_e(v(m^i)) = \{v(x^{h_j} f_j)\}$ for some $x^{h_j} f_j \in m^i$, $j = 0, \dots, e-1$.
(3) If $\sum_{j=0}^{e-1} d_j(x) f_j \in m^i$ with $d_j(x) \in W$ for all j, then $d_j(x) f_j \in m^i$ for each j.

Proof. (1)$\Rightarrow$(3): Let $a = \sum_{j=0}^{e-1} d_j(x) f_j \in m^i$. Since $\{x^{h_j} f_j\}$ is a free basis for m^i, we also have $a = \sum_{j=0}^{e-1} d'_j(x) x^{h_j} f_j$ for some $d'_j(x)$, and $d_j(x) = d'_j(x) x^{h_j}$. Now $x^{h_j} f_j \in m^i$, so $d_j(x) f_j \in m^i$.
(3)$\Rightarrow$(2): Let $u \in \mathrm{Ap}_e(v(m^i))$, so $u = v(a)$ for some $a \in m^i$. We have $a = \sum_{j=0}^{e-1} d_j(x) f_j$, with $d_j(x) f_j \in m^i$ for all j. Let $v(a) \equiv v(f_j) \pmod{e}$. Then $v(a) = v(d_j(x) f_j)$. Let $d_j(x) = \sum_{i \geq l} k_i x^i$, with $k_i \in k, k_l \neq 0$. Then we claim that $\mathrm{ord}(d_j(x) f_j) = \mathrm{ord}(x^l f_j)$. Suppose that $x^l f_j \in m^h \setminus m^{h+1}$. Then $d_j(x) f_j \in m^h$ since all summands do. If $d_j(x) f_j \in m^{h+1}$, then $k_l x^l f_j = d_j(x) f_j - \sum_{i \geq l+1} k_i x^i f_j \in m^{h+1}$, a contradiction. Thus $v(a) = v(x^l f_j)$, $x^l f_j \in m^i$.
(2)$\Rightarrow$(1): By Lemma 2.1 (1) of [2]. $\square$

Proposition 1.4 *If the m-adic filtration satisfies the BF condition, it is essentially divisible.*

Proof. Let xR be a minimal reduction of m and let $f_0, \dots, f_{e-1}$ be elements in R satisfying the BF condition, i.e. condition (2) in Proposition 1.3. We

claim that condition (2) in Proposition 1.2 is satisfied. Let $v \in v(m^i) \cap v(xR)$, $v = v_j + le$, with $v_j \in \mathrm{Ap}_e(v(m^i))$, for some $l \geq 0$. We have $v_j = v(x^{h_j} f_j)$, for some j. Thus $x^{h_j+l} f_j \in m^i \cap xR$ and $v(x^{h_j+l} f_j) = v$. Note that $h_j + l > 0$. $\square$

There are several cases in which the BF condition holds.

Proposition 1.5 *The BF-condition holds for the m-adic filtration in each of the following cases:*
(1) *R is a semigroup k-algebra.*
(2) *The reduction number r is at most 2.*
(3) *The embedding dimension ν is at most 2.*

Proof. (1): Let $R = k[[t^{n_1}, \ldots, t^{n_\nu}]]$ and $\mathrm{Ap}(v(R)) = \{w_0, \ldots, w_{e-1}\}$. Choosing the monomial Apery basis $f_j = t^{w_j}$, for $j = 0, \ldots, e-1$ and the monomial minimal reduction $xR = t^{n_1}R = t^e R$, if $\mathrm{Ap}(v(m^i)) = \{w_0 + h_0 e, \ldots, w_{e-1} + h_{e-1}e\}$, then m^i is a free $k[[t^e]]$-module generated by $t^{eh_j} f_j = t^{h_j e + w_j}$.
(2): Let xR is a minimal reduction of m and let $f_0, \ldots, f_{e-1}$ be an Apery basis of R. Then the Apery sets of $v(m^i))$, with $i \leq 2$ can always be realized as in Proposition 1.3 (2). In fact, for $v(m^2)$, note that $v(x^2 f_0) = 2e \in \mathrm{Ap}(v(m^2))$. Moreover, if $f_j \in m \setminus m^2$, then $v(xf_j) \in \mathrm{Ap}(v(m^2))$ and if $f_j \in m^2$, then $v(f_j) \in \mathrm{Ap}(v(m^2))$. If $i \geq 2$, then $m^{i+1} = xm^i$, which gives the claim.
(3) In the plane case, setting $m = \langle x, y \rangle$, using the Weierstrass Preparation Theorem, we noted in [1, Section 2] that R is a W-module generated by $1, y, y^2, ...,$ y^{e-1} and replacing each y^j with a suitable $y_j = y^j + \phi(x, y)$ ($\phi(x, y) \in m^j$), we get an Apery basis for R. Consider a power m^i of the maximal ideal. Using the above observation, m^i is generated as W-module by $x^i, x^{i-1}y, x^{i-2}y^2, \ldots, y^i, y^{i+1},$ $\ldots, y^{i(e-1)}$. Now working on the powers y^j as we do in [1], we can modify the generators, getting the e elements $x^i, x^{i-1}y, x^{i-2}y_2, \ldots, y_{e-1}$, which are still in m^i, are of the requested form and such that their values form an Apery set for $v(m^i)$. $\square$

Example Consider $R = \mathbb{C}[[t^6, t^8 + t^9]]$. Setting $x = t^6$, $y = t^8 + t^9$, as in [1], we can see that an Apery basis for R is $1, y, y_2 = y^2, y_3 = y^3 - x^4 = 3t^{25} + ..., y_4 = y^4 - x^4 y = 5t^{33} + ..., y_5 = y^5 - x^4 y^2 = 5t^{41} + ...$ Considering for example m^3, we see it is a free W-module generated by $x^3, x^2 y, xy_2, y_3, y_4, y_5$.

2 The associated graded ring

Let $\mathrm{gr}(R)$ be the associated graded ring with respect to the m-adic filtration, $\mathrm{gr}(R) = \bigoplus_{i \geq 0} m^i/m^{i+1}$. The CM-ness of $\mathrm{gr}(R)$ is equivalent to the existence of a nonzerodivisor in the homogeneous maximal ideal. If such a nonzerodivisor exists, then x^*, the image of x in $\mathrm{gr}(R)$ (where x is any element of value e) is a nonzerodivisor. We fix this notation and denote by $\mathrm{Hilb}_R(z) = \sum_{i \geq 0} l_R(m^i/m^{i+1}) z^i$ the Hilbert series of R and by $\mathrm{Hilb}_{R/xR}(z) = \sum_{i \geq 0} l_R(m^i + xR/m^{i+1} + xR) z^i$ the Hilbert series of R/xR. Recall that

$$(1 - z)\mathrm{Hilb}_R(z) \leq \mathrm{Hilb}_{R/xR}(z)$$

and the equality holds if and only if $\mathrm{gr}(R)$ is CM (cf. e.g. [3] or [4]).

We start noting that, if $\mathrm{gr}(R)$ is CM, then the conditions analyzed in the previous section are equivalent.

Proposition 2.1 *If $\mathrm{gr}(R)$ is CM, then the m-adic filtration is essentially divisible if and only if it satisfies the BF condition.*

Proof. Suppose that the m-adic filtraion is essentially divisible with respect to xR. We claim that there exist $f_0, \dots, f_{e-1}$ in R satisfying condition (2) of Proposition 1.3. If $n \geq r$, where r is the reduction number, then $m^n \subseteq xR$. Thus, if $u \in \mathrm{Ap}_e(v(m^n))$, $u \equiv j \pmod{e}$, then there exist $a \in R$, $a = xa'$, with $v(a) = u$ and $\mathrm{ord}(a) = n$. We have $v(a') = u - e$ and $\mathrm{ord}(a') = \mathrm{ord}(a) - 1$, because $\mathrm{gr}(R)$ is CM. Now there are two possibilities. If $v(a') \notin v(xR)$, i.e. $v(a') = w_j$, we choose $f_j = a'$. If $v(a') \in v(xR)$, then, since R is essentially divisible, there exist $b \in xR$, $b = xb'$, with $v(b) = v(a')$ and $\mathrm{ord}(b) = \mathrm{ord}(a')$. Moreover $b \in \mathrm{Ap}(v(m^{n-1}))$, because otherwise $u - 2e \in v(m^{n-1})$ and $u - e \in v(m^n)$, a contradiction. Continuing in this way we arrive to get the element f_j requested.

We denote by R' the first neighborhood ring or the blowup of R, i.e. the overring $\bigcup_{n \geq 0}(m^n : m^n)$. It is well known that, if $v(x) = e$, $R' = R[x^{-1}m] = \bigcup_{i \geq 0}\{yx^{-i}; y \in m^i\}$, cf. [8]. Let $w'_0, \dots, w'_{e-1}$ be the Apery set of $v(R')$ with respect to e, with $w'_j \equiv j \pmod{e}$. For each j, $j = 0, \dots, e-1$, define as in [2] a_j by $w'_j = w_j - a_j e$.

If $f_j \in m^i$, then $f_j x^{-i} \in R'$, so $v(f_j x^{-i}) = w_j - ie \in v(R')$. It follows that $w_j - b_j e \in v(R')$. Since $w'_j = w_j - a_j e$ is the smallest in $v(R')$, in its congruence class $\pmod{e}$, we have that $a_j \geq b_j$, for $j = 0, \dots, e-1$.

In [2, Theorem 2.6] we stated the following: The ring $\mathrm{gr}(R)$ is CM if and only if $a_j = b_j$, for $j = 0, \dots, e-1$.

As Lance Bryant pointed out, the proof of that theorem given in [2] works under the assumption that the m-adic filtration satisfies the BF condition.

Theorem 2.2 *If R satifies the BF condition then $\mathrm{gr}(R)$ is CM if and only if $a_j = b_j$, for $j = 0, \dots, e-1$.*

Proof. If the BF condition is satisfied, the proof given in [2] holds.

In [4] T. Cortadellas and S. Zarzuela proved, in more general hypotheses for R, a criterion for the CM-ness of $\mathrm{gr}(R)$. They consider the microinvariants of J. Elias, i.e. the numbers ϵ_j which appear in the decomposition of the torsion module

$$R'/R = \bigoplus_{j=0}^{e-1} W/x^{\epsilon_j}W$$

where R' is the blowup, xR a minimal reduction of m and $W = k[[x]]$. With our hypotheses and notation, they show in particular that $\mathrm{gr}(R)$ is CM if and only

if $c_j = \epsilon_j$, for $j = 0, \ldots, e-1$, [4, Theorem 4.2]. Comparing their result with ours, we see that they are coherent but different. In fact, if the m-adic filtration satisfies the BF condition, then, for $j = 0, \ldots, e-1$, $\epsilon_j = a_j$ by [2, Proposition 2.5] and $b_j = c_j$ by Propositions 1.2 and 1.4, so their result coincide with ours. The hypotheses on the ring in their result are more general, but the numbers c_j's depend on the minimal reduction. On the other hand, the numbers a_j's and b_j's which we consider do not depend on the minimal reduction and in our criterion the CM-ness of $\mathrm{gr}(R)$ can be read off just looking at the semigroup filtration $v(m^0) \supset v(m) \supset v(m^2) \supset \ldots$. As a matter of fact, since $R' = x^{-n}m^n$, for $n >> 0$, $v(R') = v(m^n) - ne$, for $n >> 0$, so the a_j's which relate the Apery sets of $v(R)$ and $v(R')$, can be read in the semigroup filtration $\{v(m^i)\}_{i \geq 0}$.

We give now some applications. Given an analytically irreducible ring satisfying our hypotheses, we denote by $a_j(R)$ and $b_j(R)$ the numbers defined above.

Proposition 2.3 *Let R and T be rings satifying the BF condition, with the same multiplicity e and with $a_j(R) = a_j(T)$, $b_j(R) = b_j(T)$, for $j = 0, \ldots, e-1$. If $\mathrm{gr}(R)$ is CM, then also $gr(T)$ is CM and R and T have the same Hilbert series.*

Proof. Since $\mathrm{gr}(R)$ is CM, by Theorem 2.2, $a_j(R) = b_j(R)$, for $j = 0, \ldots, e-1$. So also $a_j(T) = b_j(T)$, for $j = 0, \ldots, e-1$ and $\mathrm{gr}(T)$ is CM. If xR (respectively yT) is a minimal reduction of the maximal ideal of R (respectively of T), then, since $b_j(R) = c_j(R)$ and $b_j(T) = c_j(T)$ (cf. Proposition 1.2), the Hilbert series of R/xR and T/yT are the same. Since $\mathrm{Hilb}_{R/xR}(z) = (1-z)\mathrm{Hilb}_R(z)$ and $\mathrm{Hilb}_{T/yT}(z) = (1-z)\mathrm{Hilb}_R(z)$, also the Hilbert series of R and T are the same. $\square$

Sometimes we can use the BF condition to draw conclusions about when $\mathrm{gr}(R)$ is a complete intersection (CI). We will use that if $x \in R$ is a nonzerodivisor in R such that x^* is a nonzerodivisor in $\mathrm{gr}(R)$, then $\mathrm{gr}(R/xR) = \mathrm{gr}(R)/(x^*)$, [7, Lemma(b)].

Example If $R = k[[X,Y]]/(f)$ is a plane branch, then $\mathrm{gr}(R) = k[X,Y]/(f^*)$, where f^* is the image of f in $\mathrm{gr}(R)$, so $\mathrm{gr}(R)$ is a complete intersection. The semigroups S for which $k[[S]]$ is a CI were determined in [5]. If $\mathrm{gr}(k[[S]])$ is a CI, then necessarily $k[[S]]$ is a CI [9, Corollary 2.4]. If S is generated by three elements and is a CI, the generators are of the form $na, nb, n_1a + n_2b$, $a < b$, [6] or (with an easier proof) [10, Lemma 1]. Then

$$k[[S]] = k[[X,Y,Z]]/(X^b - Y^a, Z^n - X^{n_1}Y^{n_2})$$

It is determined in [7] when $\mathrm{gr}_m(k[[S]])$ is a CI when S is 3-generated. The result is
a) $S = \langle na, nb, n_1a \rangle$.
b) $S = \langle na, nb, n_1a + n_2b \rangle$, $na < n_1a + n_2b < nb$, $n \leq n_1 + n_2$.

c) $S = \langle na, nb, n_1a + n_2b \rangle$, $na < nb < n_1a + n_2b$, $n \le n_1 + n_2$.
Let $x = t^{na}, y = t^{nb}, z = t^{n_1a+n_2b}$.

In case a), if $n < n_1$, $\mathrm{gr}(k[[S]]/(x)) \cong k[Y, Z]/(Y^a, Z^n)$. An Apery basis for $k[[S]]$ is $\{y^i z^j; 0 \le i < a, 0 \le j < n\}$. Suppose $R = k[[t^{na}, g_2, g_3]]$ with $v(g_2) = nb, v(g_3) = n_1a$, and that $\{g_2^i g_3^j; 0 \le i < a, 0 \le j < n\}$ is an Apery basis for R, and that R satisfies the BF condition. Then $x = t^{na}$ is a minimal reduction also of the maximal ideal of R, and the a_j's and b_j's are the same for $k[[S]]$ and R, so $\mathrm{gr}(R)$ is CM, and in particular x^* is a nonzerodivisor in $\mathrm{gr}(R)$. We have that $\mathrm{gr}(R)$ is a CI if and only if $\mathrm{gr}(R/xR) = \mathrm{gr}(R)/(x^*)$ is a CI. Since $v(g_2^i g_3^j) \notin v(xR)$ if $0 \le i < a, 0 \le j < n$, and they all have values in different congruence classes (mod $v(x)$), we get that $\mathrm{gr}(R)/(x^*) \cong \mathrm{gr}(k[[S]])/(x^*) \cong k[Y, Z]/(Y^a, Z^n)$. Thus $\mathrm{gr}(R)$ is a CI. A concrete example is $R = k[[t^6, t^8 + ct^{13} + dt^{19}, t^9]]$, $c, d \in k$.

If $n_1 < n$, then $\mathrm{gr}(k[[S]]/(z)) = k[X, Y]/(Y^a, X^{n_1})$, and $\{y^i x^j; 0 \le i < a, 0 \le j < n_1\}$ is an Apery basis for $k[[S]]$. Suppose $R = k[[t^{n_1a}, g_2, g_3]]$ with $v(g_2) = na, v(g_3) = nb$, and that $\{g_3^i g_2^j; 0 \le i < a, 0 \le j < n_1\}$ is an Apery basis for R, and that R satisfies the BF condition. As above we get that $\mathrm{gr}(R)$ is a CI. A concrete example is $k[[t^6, t^9 + ct^{11}, t^4]]$, $c \in k$.

In case b) an Apery set is $\{y^i z^j; 0 \le i < a, 0 \le j < n\}$. Suppose $R = k[[t^{na}, g_2, g_3]]$, $v(g_2) = n_1a + n_2b, v(g_3) = nb$, and that $\{g_3^i g_2^j; 0 \le i < a, 0 \le j < n\}$ is an Apery set for R, and that R satifies the BF condition. Reasoning as above, we get that $\mathrm{gr}(R)$ is a CI. A concrete example is $k[[t^6, t^7 + ct^{11}, t^9]]$, $c \in k$.

In case c) an Apery set is $\{y^i z^j; 0 \le i < a, 0 \le j < n\}$. Suppose $R = k[[t^{na}, g_2, g_3]]$, $v(g_2) = nb, v(g_3) = n_1a + n_2b$, and that $\{g_2^i g_3^j; 0 \le i < a, 0 \le j < n\}$ is an Apery set for R, and that R satifies the BF condition. Reasoning as above, we get that $\mathrm{gr}(R)$ is a CI. A concrete example is $k[[t^4, t^6, t^7 + ct^9]]$, $c \in k$.

We end with some questions:

1. Does the converse of Proposition 1.4 hold?

2. Is Theorem 2.2 true, without assuming the BF-condition?

3. Is always $\epsilon_j = a_j$, for $j = 0, \ldots, e - 1$ without assuming the BF-condition?

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